Exact soliton solutions and nonlinear modulation instability in spinor Bose-Einstein condensates

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Abstract

We find one-, two-, and three-component solitons of the polar and ferromagnetic (FM) types in the general (non-integrable) model of a spinor (three-component) model of the Bose-Einstein condensate (BEC), based on a system of three nonlinearly coupled Gross-Pitaevskii equations. The stability of the solitons is studied by means of direct simulations, and, in a part, analytically, using linearized equations for small perturbations. Global stability of the solitons is considered by means of the energy comparison. As a result, ground-state and metastable soliton states of the FM and polar types are identified. For the special integrable version of the model, we develop the Darboux transformation (DT). As an application of the DT, analytical solutions are obtained that display full nonlinear evolution of the modulational instability (MI) of a continuous-wave (CW) state seeded by a small spatially periodic perturbation. Additionally, by dint of direct simulations, we demonstrate that solitons of both the polar and FM types, found in the integrable system, are structurally stable, i.e., they are robust under random changes of the relevant nonlinear coefficient in time.

I. INTRODUCTION

The experimental observation of matter-wave solitons of the dark [1], bright [2], and gap [3] types in Bose-Einstein condensates (BECs) have attracted a great deal of attention to the dynamics of nonlinear matter waves, including such aspects as the soliton propagation [4], vortex dynamics [5], interference patterns [6], domain walls in binary BECs [7], and the modulational instability (MI) [8, 9].

One of major developments in BECs was the study of spinor condensates. Spinor BECs feature an intrinsic three-component structure, due to the distinction between different hyperfine spin states of the atoms. When spinor BECs are trapped in the magnetic potential, the spin degree of freedom is frozen. However, in the condensate held by an optical potential, the spin is free, making it possible to observe a rich variety of phenomena, such as spin domains [10] and textures [11]. Recently, properties of BECs with this degree of freedom were investigated in detail [12] - [22], experimentally and theoretically. In particular, the MI of a continuous-wave (CW) state with constant densities in all the three components (sometimes, it is called "driven current"), was explored in Ref. [18]. An important results was reported in Ref. [22], which demonstrated that, under special constraints imposed on parameters, the matrix nonlinear Schrödinger (NLS) equation, which is a model of the one-dimensional (1D) spinor BEC in the free space, may be integrable by means of the inverse scattering transform. For that case, exact single-soliton solutions, as well as solutions describing collisions between two solitons, were found.

In this work, we aim to consider several species of simple one-, two-, and three-component soliton solutions, of both the ferromagnetic (FM) and polar types, which can be found in an analytical form in the general (non-integrable) version of the spinor model. Dynamical stability of the solitons will be studied by means of analytical and numerical methods, and global stability of different soliton types will be considered by comparison of their energies, which will allow us to identify ground-state and metastable soliton states. For the special integrable version of the model, we will present a Darboux transformation (DT) and a family of exact solutions generated by it from CW states. If the amplitude of the CW background vanishes, these solutions go over into solitons found in Ref. [22]. The family also contains new spatially periodic time-dependent solutions, that explicitly describe the full nonlinear development of the MI, starting from a CW state seeded with an infinitely small spatially

periodic perturbation. In addition, we demonstrate structural stability of the solitons of both polar and FM types in the integrable model, by showing that they are robust under random changes of a nonlinear coefficient whose value determines the integrability.

The paper is organized as follows. In Section 2, the model is formulated. Various species of exact soliton solutions in the non-integrable model, and their stability, are considered in Section 3. Section 4 introduces the DT, presents solutions obtained by means of this technique, and also numerical results demonstrating the structural stability of the solitons. The paper is concluded by Section 5.

II. THE MODEL

We start with the consideration of an effectively one-dimensional BEC trapped in a pencil-shaped region, which is elongated in x and tightly confined in the transverse directions y, z (see, e.g., Ref. [9]). Therefore, atoms in the $F = 1$ hyperfine state can be described by a 1D vectorial wave function, $\mathbf{\Phi}(x,t) = [\Phi_{+1}(x,t), \Phi_0(x,t), \Phi_{-1}(x,t)]^T$, with the components corresponding to the three values of the vertical spin projection, $m_F = +1, 0, -1$. The wave functions obey a system of coupled Gross-Pitaevskii (alias NLS) equations [22],

$$\begin{aligned} i\hbar\partial_t\Phi_{\pm 1} &= -\frac{\hbar^2}{2m}\partial_x^2\Phi_{\pm 1} + (c_0 + c_2)(|\Phi_{\pm 1}|^2 + |\Phi_0|^2)\Phi_{\pm 1} \\ &+ (c_0 - c_2)|\Phi_{\mp 1}|^2\Phi_{\pm 1} + c_2\Phi_{\mp 1}^*\Phi_0^2, \\ i\hbar\partial_t\Phi_0 &= -\frac{\hbar^2}{2m}\partial_x^2\Phi_0 + (c_0 + c_2)(|\Phi_{+1}|^2 + |\Phi_{-1}|^2)\Phi_0 \\ &+ c_0|\Phi_0|^2\Phi_0 + 2c_2\Phi_{+1}\Phi_{-1}\Phi_0^*, \end{aligned} \tag{1}$$

where $c_0 = (g_0+2g_2)/3$ and $c_2 = (g_2-g_0)/3$ denote effective constants of the mean-field (spin-independent) and spin-exchange interaction, respectively [14]. Here $g_f = 4\hbar^2 a_f/[m a_\perp^2(1 - c a_f/a_\perp)]$, with $f = 0, 2$, are effective 1D coupling constants, a_f is the s-wave scattering length in the channel with the total hyperfine spin f, $a_\perp$ is the size of the transverse ground state, m is the atomic mass, and $c = -\zeta(1/2) \approx 1.46$. Redefining the wave function as $\mathbf{\Phi} \to (\phi_{+1}, \sqrt{2}\phi_0, \phi_{-1})^T$ and measuring time and length in units of $\hbar/|c_0|$ and $\sqrt{\hbar^2/2m|c_0|}$,

respectively, we cast Eqs. (1) in a normalized form

$$
\begin{aligned}
i\partial_t\phi_{\pm 1} &= -\partial_x^2\phi_{\pm 1} - (\nu + a)(|\phi_{\pm 1}|^2 + 2|\phi_0|^2)\phi_{\pm 1} \\
&\quad - (\nu - a)|\phi_{\mp 1}|^2\phi_{\pm 1} - 2a\phi_{\mp 1}^*\phi_0^2, \\
i\partial_t\phi_0 &= -\partial_x^2\phi_0 - 2\nu|\phi_0|^2\phi_0 \\
&\quad - (\nu + a)(|\phi_{+1}|^2 + |\phi_{-1}|^2)\phi_0 - 2a\phi_{+1}\phi_{-1}\phi_0^*,
\end{aligned} \tag{2}
$$

where $\nu \equiv -\mathrm{sgn}\,(c_0)$, $a \equiv -c_2/|c_0|$.

These equations can be derived from the Hamiltonian

$$
\begin{aligned}
H = \int_{-\infty}^{+\infty} dx \Big\{ &\left|\partial_x\phi_{+1}\right|^2 + \left|\partial_x\phi_{-1}\right|^2 - \frac{1}{2}(\nu + a)\left(|\phi_{+1}|^4 + |\phi_{-1}|^4\right) - (\nu - a)|\phi_{+1}|^2|\phi_{-1}|^2 \\
&+2\left[|\partial_x\phi_0|^2 - \nu\,|\phi_0|^4 - (\nu + a)\left(|\phi_{+1}|^2 + |\phi_{-1}|^2\right)|\phi_0|^2 - a\left(\phi_{+1}^*\phi_{-1}^*\phi_0^2 + \phi_{+1}\phi_{-1}\left(\phi_0^*\right)^2\right)\right]\Big\}\,,
\end{aligned} \tag{3}
$$

which is a dynamical invariant of the model ($dH/dt = 0$). In addition, Eqs. (2) conserve the momentum,

$$
P = i\int_{-\infty}^{+\infty}\left(\phi_{+1}^*\partial_x\phi_{+1} + \phi_{-1}^*\partial_x\phi_{-1} + 2\phi_0^*\partial_x\phi_0\right)dx,
$$

and the solution's norm, proportional to total number of atoms,

$$
N = \int_{-\infty}^{+\infty}\left[\left|\phi_{+1}(x,t)\right|^2 + \left|\phi_{-1}(x,t)\right|^2 + 2\left|\phi_0(x,t)\right|^2\right]dx. \tag{4}
$$

An obvious reduction of equations (2) can be obtained by setting $\phi_0 \equiv 0$. In this case, the model reduces to a system of two equations which are tantamount to those describing light transmission in bimodal nonlinear optical fibers, with the two modes representing either different wavelengths or two orthogonal polarizations [23]:

$$
i\partial_t\phi_{\pm 1} = -\partial_x^2\phi_{\pm 1} - (\nu + a)|\phi_{\pm 1}|^2\phi_{\pm 1} - (\nu - a)|\phi_{\mp 1}|^2\phi_{\pm 1}. \tag{5}
$$

In particular, the case of two linear polarizations in a birefringent fiber corresponds to $(\nu - a)/(\nu + a) = 2/3$, i.e., $a = \nu/5$, and two different carrier waves or circular polarizations correspond to $(\nu - a)/(\nu + a) = 2$, i.e., $a = -\nu/3$. The two nonlinear terms in Eqs. (5) account for, respectively, the SPM and XPM (self- and cross-phase-modulation) interactions of the two waves. The MI of CW (uniform) states in the system of two XPM-coupled equations (5) has been studied in detail [23]. If the nonlinearity in Eqs. (5) is self-defocusing, i.e., $\nu = -1$ and $1 \pm a > 0$(in other words, $|a| < 1$), the single NLS equation would of course show no MI, but the XPM-coupled system of NLS equations (5) gives rise to MI, provided that the XPM interaction is stronger than SPM, i.e., $0 < a < 1$.

III. EXACT SINGLE-, TWO-, AND THREE-COMPONENT SOLITON SOLUTIONS

The previous consideration of solitons in the model of the spinor BEC was focused on the particular integrable case, $a = \nu = 1$ [22]. As follows from the above derivation, this special case is physically possible, corresponding to $c_2 = c_0$, or, equivalently, $2g_0 = -g_2 > 0$. The latter condition can be satisfied by imposing a condition on the scattering lengths,

$$a_\perp = 3ca_0a_2/(2a_0 + a_2), \tag{6}$$

provided that $a_0a_2(a_2 - a_0) > 0$ holds.

However, it is also possible to find simple exact soliton solutions of both polar and FM types in the general case, $a \neq \nu$. Below, we present several soliton species for this case, and study their stability.

A. Single-component ferromagnetic soliton

First, a single-component FM soliton is given by a straightforward solution (provided that $a + \nu > 0$),

$$\phi_{-1} = \phi_0 = 0,\ \phi_{+1} = \sqrt{\frac{-2\mu}{\nu + a}}\mathrm{sech}\left(\sqrt{-\mu}x\right)e^{-i\mu t}, \tag{7}$$

where the negative chemical potential μ is the intrinsic parameter of the soliton family. While the expression (7) corresponds to a zero-velocity soliton, moving ones can be generated from it in an obvious way by means of the Galilean transformation. Note that the condition $a + \nu > 0$ implies $a > 1$ or $a > -1$ in the cases of the, respectively, repulsive ($\nu = -1$) and attractive ($\nu = +1$) spin-independent interaction. The norm (4) and energy (3) of this soliton are

$$N = \frac{4\sqrt{-\mu}}{\nu + a},\ H = -\frac{(\nu + a)^2}{48}N^3. \tag{8}$$

The soliton (7) is *stable* against infinitesimal perturbations, as the linearization of Eqs. (2) about this solution demonstrates that all the three equations for small perturbations are *decoupled*. Then, as the standard soliton solution of the single NLS equation is always stable, solution (7) cannot be unstable against small perturbations of ϕ_{+1}. Further, the decoupled

linearized equations for small perturbations ϕ_{-1} and ϕ_0 of the other fields are

$$i\partial_t\phi_{-1} = -\partial_x^2\phi_{-1} - (\nu - a)|\phi_{+1}|^2\phi_{-1}, \tag{9}$$

$$i\partial_t\phi_0 = -\partial_x^2\phi_0 - (\nu + a)|\phi_{+1}|^2\phi_0, \tag{10}$$

and it is well known that such linear equations, with $|\phi_{+1}|^2$ corresponding to the unperturbed soliton (7), give rise to no instabilities either.

B. Single-component polar soliton

The simplest polar soliton, that has only the ϕ_0 component, can be found for $\nu = +1$ (note that the solution does not depend on the other nonlinear coefficient a):

$$\phi_0 = \sqrt{-\mu}\mathrm{sech}\left(\sqrt{-\mu}x\right)e^{-i\mu t},\ \ \phi_{\pm1} = 0. \tag{11}$$

The stability problem for this soliton is more involved, as the unperturbed field ϕ_0 gives rise to a *coupled* system of linearized equations for infinitesimal perturbations $\phi_{\pm1}$ of the other fields:

$$\left(i\partial_t + \mu\right)\chi_{\pm1} = -\partial_x^2\chi_{\pm1} + 2\mu\mathrm{sech}^2\left(\sqrt{-\mu}x\right)\left[(1+a)\chi_{\pm1} + a\chi_{\mp1}^*\right], \tag{12}$$

where we have substituted $\nu = +1$ and redefined the perturbation,

$$\phi_{\pm1} \equiv \exp\left(-i\mu t\right)\chi_{\pm1}. \tag{13}$$

The last term in Eqs. (12) gives rise to a *parametric gain*, which may be a well-known source of instability (see, e.g., Ref. [24]). This instability can be easily understood in qualitative terms if one replaces Eqs. (12) by simplified equations, that disregard the x-dependence, and replace the waveform $\mathrm{sech}^2\left(\sqrt{-\mu}x\right)$ by its value at the soliton's center, $x = 0$:

$$i\frac{d\chi_{\pm1}}{dt} = \mu\left[(1+2a)\chi_{\pm1} + 2a\chi_{\mp1}^*\right]. \tag{14}$$

An elementary consideration shows that the zero solution of linear equations (14) is double-unstable (i.e., through a double eigenvalue) in the region of $a < -1/4$.

We checked the stability of the single-component polar soliton (11) by means of direct simulations of Eqs. (2), adding a small random perturbation in the components ϕ_{+1} and ϕ_{-1}, the value of the perturbation being distributed uniformly between 0 and 0.03 (the same random perturbation will be used in simulations of the stability of other solitons, see below).

The result is that the soliton (11) is always unstable indeed, as an example in Fig. 1. In particular, panel (d) in this figure displays the time evolution of

$$N_0 = \int_{-\infty}^{+\infty} |\phi_0(x,t)|^2 \, dx, N_{\pm 1} = \int_{-\infty}^{+\infty} \left|\phi_{\pm}(x,t)\right|^2 dx. \tag{15}$$

C. Two-component polar soliton

In the same case as considered above, $\nu = +1$, a two-component polar soliton can be easily found too:

$$\phi_0 = 0, \;\; \phi_{+1} = \pm \phi_{-1} = \sqrt{-\mu} \mathrm{sech} \left(\sqrt{-\mu} x\right) e^{-i\mu t} \tag{16}$$

(note that it again does depend on the value of the nonlinearity coefficient a). In this case again, instability is possible due to the parametric gain in linearized equations for small perturbations. Indeed, the corresponding equation for the perturbation in the ϕ_0 component decouples from the other equations and takes the form

$$\left(i\partial_t + \mu\right)\chi_0 = -\partial_x^2 \chi_0 + 2\mu \; \mathrm{sech}^2 \left(\sqrt{-\mu} x\right) \left[(1+a)\chi_0 \pm a \chi_0^*\right], \tag{17}$$

where the perturbation was redefined the same way as in Eq. (13), $\phi_0 \equiv \exp\left(-i\mu t\right) \chi_0$. Similar to the case of Eq. (12), the origin of the instability may be understood, replacing Eq. (17) by its simplified version that disregards the x-dependence and substitutes the waveform $\mathrm{sech}^2 \left(\sqrt{-\mu} x\right)$ by its value at the central point of the soliton, $x = 0$:

$$i \frac{d\chi_0}{dt} = \mu \left[(1+2a)\chi_0 \pm 2a\chi_0^*\right], \tag{18}$$

cf. Eqs. (14). An elementary consideration demonstrates that the zero solution of Eq. (18) (for either sign $\pm$) is unstable in exactly the same region as in the case of Eqs. (14), $a < -1/4$. Note that the simplified equation (18) illustrates only the qualitative mechanism of the parametric instability, while the actual stability border may be different from $a = -1/4$. Indeed, the stability of the soliton (16) was tested by direct simulations of Eqs. (2) with a small uniformly distributed random perturbation added to the ϕ_0 component. The result shows that the soliton is indeed unstable in the region of $|a| \;\geq 1$, as shown in Fig. 2, and it is stable if $|a| < 1$ (not shown here).

The change of the stability of this soliton at $a = +1$ can be explained analytically. Indeed, splitting the perturbation into real and imaginary parts, $\chi_0(x,t) \equiv \chi_1(x,t) + i\chi_2(x,t)$, and

looking for a perturbation eigenmode as $\chi_{1,2}(x,t) = e^{\sigma t} U_{1,2}(x)$ with the instability growth rate (eigenvalue) σ, one arrives at a system of real equations [take plus sign in (17)],

$$-\sigma U_2 = -U_1'' - \mu U_1 + 2\mu(1+2a)\mathrm{sech}^2\left(\sqrt{-\mu}x\right) U_1,$$
$$\sigma U_1 = -U_2'' - \mu U_2 + 2\mu\ \mathrm{sech}^2\left(\sqrt{-\mu}x\right) U_2, \tag{19}$$

with prime standing for d/dx. The simplest possibility for the onset of instability of the soliton is the passage of σ from negative values, corresponding to neutral stability, to unstable positive values. At the critical (zero-crossing) point, $\sigma = 0$, equations (19) decouple, and each one becomes explicitly solvable, as is commonly known from quantum mechanics. The exact solution of the decoupled equations shows that σ crosses zero at points $a = a_n \equiv n(n+3)/4$, $n = 0, 1, 2, \ldots$. The vanishing of σ at the point $a_0 = 0$ corresponds not to destabilization of the soliton, but just to the fact that Eqs. (19) become symmetric at this point, while the zero crossings at other points indeed imply stability changes. In particular, the critical point $a_1 = 1$ explains the destabilization of soliton (16) at $a = 1$, as observed in the simulations, the corresponding eigenfunctions being $U_1 = \sinh\left(\sqrt{-\mu}x\right) \cdot \mathrm{sech}^2\left(\sqrt{-\mu}x\right)$, $U_2 = \mathrm{sech}\left(\sqrt{-\mu}x\right)$. Critical points corresponding to $n > 1$ imply, most probably, additional destabilizations (the emergence of new unstable eigenmodes) of the already unstable soliton. On the other hand, the destabilization of the soliton at $a = -1$, also observed in the simulations, may be explained by a more involved bifurcation, when a pair of the eigenvalues σ change from purely imaginary to complex ones, developing an unstable (positive) real part (a generic route to the instability of the latter type is accounted for by the *Hamiltonian Hopf bifurcation*, which involves a collision between two pairs of imaginary eigenvalues, and their transformation into an unstable quartet [25]).

D. Three-component polar solitons

In the case of $\nu = +1$, three-component solitons of the polar type can be found too. One of them is

$$\phi_0 = \sqrt{1-\epsilon^2}\sqrt{-\mu}\ \mathrm{sech}\left(\sqrt{-\mu}x\right) e^{-i\mu t},$$
$$\phi_{+1} = -\phi_{-1} = \pm\epsilon\sqrt{-\mu}\ \mathrm{sech}\left(\sqrt{-\mu}x\right) e^{-i\mu t}, \tag{20}$$

where ϵ is an arbitrary parameter taking values $-1 < \epsilon < +1$ (the presence of this parameter resembles the feature typical to solitons in the Manakov's system [23, 26]), and, as well as

the one- and two-component polar solitons displayed above, the solution does not explicitly depend on the parameter a. We stress that the phase difference of π between the ϕ_{+1} and ϕ_{-1} components is a necessary ingredient of the solution.

There is another three-component polar solution similar to the above one (i.e., containing the arbitrary parameter ϵ, and independent of a), but with equal phases of the $\phi_{\pm 1}$ components and a phase shift of $\pi/2$ in the ϕ_0 component. This solution is

$$\begin{aligned} \phi_0 &= i\sqrt{1-\epsilon^2}\sqrt{-\mu}\ \mathrm{sech}\left(\sqrt{-\mu}x\right)e^{-i\mu t}, \\ \phi_{+1} &= \phi_{-1} = \pm\epsilon\sqrt{-\mu}\ \mathrm{sech}\left(\sqrt{-\mu}x\right)e^{-i\mu t}, \end{aligned} \tag{21}$$

where the sign $\pm$ is the same for both components.

In addition, there is a species of three-component polar solitons that explicitly depend on a:

$$\begin{aligned} \phi_0 &= \left(\mu_{+1}\mu_{-1}\right)^{1/4}\ \mathrm{sech}\left(\sqrt{-\mu}x\right)e^{-i\mu t}, \\ \phi_{\pm 1} &= \sqrt{-\mu_{\pm 1}}\ \mathrm{sech}\left(\sqrt{-\mu}x\right)e^{-i\mu t}, \end{aligned} \tag{22}$$

where $\mu_{\pm 1}$ are two arbitrary negative parameters, and the chemical potential is $\mu = -\left(\nu+a\right)\left(\sqrt{-\mu_{+1}}+\sqrt{-\mu_{-1}}\right)^2/2$, which implies that $\nu + a > 0$ (note that this solution admits $v = -1$, i.e., repulsive spin-independent interaction). Each species of the three-component polar soliton depends on two arbitrary parameters: either μ and ϵ [solutions (20) and (21)], or μ_{-1} and μ_{+1} [solution (22)].

The stability of all the species of these solitons was tested in direct simulations. In the case of solutions (20) and (21), it was enough to seed a small random perturbation only in the ϕ_0 component, to observe that both these types are unstable, as shown in Fig. 3 and 4.

On the contrary, the three-component polar soliton (22) is *completely stable*. This point was checked in detail, by seeding small random perturbations in all the components. The result, illustrated by Fig. 5, is that the perturbation induces only small oscillations of amplitudes of each component of the soliton, and remain robust as long as the simulations were run.

E. Global stability of the solitons

It follows from the above results that the model demonstrates remarkable multistability, as the FM soliton (7), the two-component polar solitons (16) (in the regions of $-1 < a < +1$),

and the three-component soliton (22) may all be stable in one and the same parameter region. Then, to identify which solitons are "more stable" and "less stable", one may fix the norm (4) and compare respective values of the Hamiltonian (3) for these solutions, as the ground-state solution should correspond to a minimum of H at given N.

The substitution of the solutions in Eq. (3) reveals another remarkable fact: all the solutions which do not explicitly depend on a, i.e., the one-, two-, and three-component polar solitons (11), (16), and (20), (21), produce identical relations between the chemical potential (μ), norm, and Hamiltonian:

$$N = 4\sqrt{-\mu}, \; H = -\frac{1}{48}N^3. \tag{23}$$

As mentioned above, these relations are different for the single-component FM soliton (7), see Eqs. (8). Finally, for the stable three-component polar soliton (22), the relations between μ, N, and H take exactly the same form as Eqs. (8) for the FM soliton.

The comparison of the expressions (23) and (8) leads to a simple conclusion: the single-component FM soliton (7) and the stable three-component polar soliton (22) *simultaneously* provide for the minimum of energy in the case of $\nu = +1$ and $a > 0$, i.e., when both the spin-independent and spin-exchange interactions between atoms are attractive (recall a necessary condition for the existence of both these types of the solitons is $\nu + a > 0$). It is possible to state that, in these cases, each of these two species, (7) and (22), plays the role of the ground state in its own class of the solitons (ferromagnetic and polar ones, respectively). Simultaneously, the two-component soliton (16) is also stable in the region of $0 < a < 1$, as shown above, but it corresponds to higher energy, hence it represents a metastable state in this region.

The FM soliton (7) and three-component polar soliton (22) also exist in the case of $\nu = -1$ and $a > 1$, when $\nu + a$ is positive. In this case too, these two soliton species provide for the energy minimum, simply because the other solitons, (11), (16), (20), and (21) do not exist at $\nu = -1$.

For $\nu = +1$ and $-1 < a < 0$ (attractive spin-independent and repulsive spin-exchange interactions), the Hamiltonian value (23) is smaller than the competing one (8). Actually, this means that the two-component polar soliton (16) plays the role of the ground state in this case, as only it, among all the polar solitons whose Hamiltonian is given by the expression (23), is dynamically stable (for $-1 < a < +1$, see above). As the FM soliton

(7) and three-component polar soliton (22) also exist and are stable in this region, but correspond to greater energy, they are metastable states here. Finally, for $\nu = +1$ and $a < -1$, there are no stable solitons (among the ones considered above).

The dependence $N(\mu)$ for each solution family provides additional concerning the soliton stability. Indeed, the known Vakhitov-Kolokolov (VK) criterion [27] states that a necessary stability condition for the soliton family is $dM/d\mu < 0$. It guarantees that the soliton cannot be unstable against perturbations with real eigenvalues, but does not say anything about oscillatory perturbation modes appertaining to complex eigenvalues. Obviously, both relations, given by Eqs. (8) and (23), satisfy the VK condition, i.e., any unstable soliton may be unstable only against perturbations that grow in time with oscillations. Indeed, numerical results which illustrate the instability of those solitons which are not stable, see Figs. 1-4, clearly suggest that the instability, if any, is oscillatory.

F. Finite-background solitons

In special cases, it is possible to find exact solutions for solitons sitting on a nonzero background. Namely, for $\nu = 1$ and $a = -1/2$, one can find a two-component polar soliton with a continuous-wave (CW) background attached to it, in the following form: $\phi_0 = 0$, and

$$\begin{aligned} \phi_{+1} &= e^{-i\mu t}\sqrt{-\mu}\left[\frac{1}{\sqrt{2}} \pm \operatorname{sech}\left(\sqrt{-\mu}x\right)\right], \\ \phi_{-1} &= e^{-i\mu t}\sqrt{-\mu}\left[\frac{1}{\sqrt{2}} \mp \operatorname{sech}\left(\sqrt{-\mu}x\right)\right]. \end{aligned} \tag{24}$$

For $\nu = a = 1$, a three-component polar solution with the background can be found too:

$$\begin{aligned} \phi_{+1} = \phi_{-1} &= \frac{1}{2}\sqrt{-\mu}e^{-i\mu t}\left[\frac{1}{\sqrt{2}} \pm \operatorname{sech}\left(\sqrt{-\mu}x\right)\right], \\ \phi_0 &= \frac{1}{2}\sqrt{-\mu}e^{-i\mu t}\left[\frac{1}{\sqrt{2}} \mp \operatorname{sech}\left(\sqrt{-\mu}x\right)\right] \end{aligned} \tag{25}$$

(in the latter case, the availability of the exact solution is not surprising, as the case of $\nu = a = 1$ is the exactly integrable one [22]).

The stability of solutions (24) and (25) was tested, simultaneously perturbing the ϕ_0 and $\phi_{\pm 1}$ components by small uniformly distributed random perturbations. Figures 6 and 7 show that both solutions are unstable, although the character of the instability is different: in the former case, the soliton itself (its core) seems stable, while the background appears to

be modulationally unstable. In the latter case (Fig. 7), the background is modulationally stable, but the core of the soliton (25) clearly features an oscillatory instability.

The modulational instability (MI) of the background in solution (24) is obvious, as, even without exciting the ϕ_0 field, i.e., within the framework of the reduced equations (5), with $\nu + a = 1/2$ and $\nu - a = 3/2$, any CW solution is obviously subject to MI. As concerns the excitation of the ϕ_0 field, an elementary consideration shows that the CW part of solution (24) exactly corresponds to the threshold of the instability possible through the action of the parametric-drive term, $2a\phi_{+1}\phi_{-1}\phi_0^*$, in the last equation (2). Actually, Fig. 6 demonstrate that this parametric instability sets in, which may be explained by conjecturing that the initial development of the above-mentioned MI that does not include the ϕ_0 field drives the perturbed system across the threshold of the parametric instability.

IV. THE DARBOUX TRANSFORMATION AND NONLINEAR DEVELOPMENT OF THE MODULATIONAL INSTABILITY

In this section, we focus on the integrable case, with $\nu = a = 1$, which corresponds to the attractive interactions. As explained above, the spinor BEC obeys this condition if a special (but physically possible) constraint (6) is imposed on the scattering lengths which determine collisions between atoms. Then, Eqs. (2) can be rewritten as a 2×2 matrix NLS equation [22, 28],

$$i\partial_t \mathbf{Q} + \partial_x^2 \mathbf{Q} + 2\mathbf{Q}\mathbf{Q}^\dagger \mathbf{Q} = 0, \ \mathbf{Q} \equiv \begin{pmatrix} \phi_{+1} & \phi_0 \\ \phi_0 & \phi_{-1} \end{pmatrix}. \tag{26}$$

Equation (26) is a completely integrable system, to which the inverse scattering transform applies [28] [in particular, with regard to the fact that $\nu - a = 0$ in the present case, Eqs. (5) show that system (26) splits into two uncoupled NLS equations for solutions with $\phi_0 = 0$]. We here aim to develop the Darboux transformation (DT) for Eqs. (26), based on the respective Lax pair.

The Lax-pair representation of Eqs. (26) is [28]

$$\mathbf{\Psi}_x = \mathbf{U}\mathbf{\Psi}, \ \mathbf{\Psi}_t = \mathbf{V}\mathbf{\Psi}, \tag{27}$$

where $\mathbf{U} = \lambda\mathbf{J} + \mathbf{P}$, $\mathbf{V} = i2\lambda^2\mathbf{J} + i2\lambda\mathbf{P} + i\mathbf{W}$, with

$$\mathbf{J} = \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I} \end{pmatrix}, \mathbf{P} = \begin{pmatrix} \mathbf{O} & \mathbf{Q} \\ -\mathbf{Q}^\dagger & \mathbf{O} \end{pmatrix}, \tag{28}$$

$$\mathbf{W} = \begin{pmatrix} \mathbf{Q}\mathbf{Q}^\dagger & \mathbf{Q}_x \\ \mathbf{Q}_x^\dagger & -\mathbf{Q}^\dagger\mathbf{Q} \end{pmatrix}. \tag{29}$$

Here $\mathbf{I}$ denotes the 2×2 unit matrix, $\mathbf{O}$ is the 2×2 zero matrix, $\mathbf{\Psi} = \begin{pmatrix} \mathbf{\Psi}_1 \\ \mathbf{\Psi}_2 \end{pmatrix}$ is the matrix eigenfunction corresponding to λ, $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_2$ being 2×2 matrices, and λ is the spectral parameter. Accordingly, Eq. (26) can be recovered from the compatibility condition of the linear system (27), $\mathbf{U}_t - \mathbf{V}_x + \mathbf{U}\mathbf{V} - \mathbf{V}\mathbf{U} = 0$.

Next, based on the Lax pair (27), and introducing a transformation in the form

$$\widetilde{\mathbf{\Psi}} = (\lambda - \mathbf{S})\mathbf{\Psi},\ \mathbf{S} = \mathbf{D}\mathbf{\Lambda}\mathbf{D}^{-1},\ \mathbf{\Lambda} = \begin{pmatrix} \lambda_1\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \lambda_2\mathbf{I} \end{pmatrix}, \tag{30}$$

where $\mathbf{D}$ is a nonsingular matrix that satisfies $\mathbf{D}_x = \mathbf{J}\mathbf{D}\mathbf{\Lambda} + \mathbf{P}\mathbf{D}$, and letting

$$\widetilde{\mathbf{\Psi}}_x = \widetilde{\mathbf{U}}\widetilde{\mathbf{\Psi}},\ \widetilde{\mathbf{U}} = \lambda\mathbf{J} + \mathbf{P}_1,\ \mathbf{P}_1 \equiv \begin{pmatrix} \mathbf{O} & \mathbf{Q}_1 \\ -\mathbf{Q}_1^\dagger & \mathbf{O} \end{pmatrix}, \tag{31}$$

one obtains

$$\mathbf{P}_1 = \mathbf{P} + \mathbf{J}\mathbf{S} - \mathbf{S}\mathbf{J}. \tag{32}$$

Also, one can verify the following *involution* property of the above linear equations: if $\mathbf{\Psi} = \begin{pmatrix} \mathbf{\Psi}_1 \\ \mathbf{\Psi}_2 \end{pmatrix}$ is an eigenfunction corresponding to λ, where $\mathbf{\Psi}_j$ are 2×2 matrices, then $\begin{pmatrix} -\mathbf{\Psi}_2^* \\ \mathbf{\Psi}_1^* \end{pmatrix}$ is an eigenfunction corresponding to $-\lambda^*$, where we have used the fact that $\mathbf{Q}^* = \mathbf{Q}^\dagger$ [the matrix $\mathbf{Q}$ is symmetric, as per its definition (26)]. Thus we can take $\mathbf{D}$ in the form

$$\mathbf{D} = \begin{pmatrix} \mathbf{\Psi}_1 & -\mathbf{\Psi}_2^* \\ \mathbf{\Psi}_2 & \mathbf{\Psi}_1^* \end{pmatrix},\ \mathbf{\Lambda} = \begin{pmatrix} \lambda\mathbf{I} & \mathbf{O} \\ \mathbf{O} & -\lambda^*\mathbf{I} \end{pmatrix},$$

to obtain

$$\mathbf{S} = \lambda \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} + (\lambda + \lambda^*) \begin{pmatrix} -\mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & -\mathbf{S}_{22} \end{pmatrix},$$

where the matrix elements of $\mathbf{S}$ are given by $\mathbf{S}_{11} = (\mathbf{I} + \mathbf{\Psi}_1\mathbf{\Psi}_2^{-1}\mathbf{\Psi}_1^*\mathbf{\Psi}_2^{*-1})^{-1}$, $\mathbf{S}_{12} = (\mathbf{\Psi}_2\mathbf{\Psi}_1^{-1} + \mathbf{\Psi}_1^*\mathbf{\Psi}_2^{*-1})^{-1}$, $\mathbf{S}_{21} = (\mathbf{\Psi}_2^*\mathbf{\Psi}_1^{*-1} + \mathbf{\Psi}_1\mathbf{\Psi}_2^{-1})^{-1}$, $\mathbf{S}_{22} = (\mathbf{I} + \mathbf{\Psi}_2\mathbf{\Psi}_1^{-1}\mathbf{\Psi}_2^*\mathbf{\Psi}_1^{*-1})^{-1}$.

Finally, the DT for Eq. (26) follows from relation (32):

$$\mathbf{Q}_1 = \mathbf{Q} + 2(\lambda + \lambda^*)\mathbf{S}_{12}. \tag{33}$$

It should be noted that $\mathbf{S}_{21}^* = \mathbf{S}_{12}$ implies the above-mentioned property, $\mathbf{Q}_1^* = \mathbf{Q}_1^\dagger$. From Eqs. (33) and (27), it can be deduced that Eq. (33) generates a new solution $\mathbf{Q}_1$ for Eq. (26) once a *seed solution* $\mathbf{Q}$ is known. In particular, a one-soliton solution can be generated if the seed is a trivial (zero) state. Next, taking $\mathbf{Q}_1$ as the new seed solution, one can derive from Eq. (33) the corresponding two-soliton solution. This can be continued as a recursion procedure generating multi-soliton solutions.

In the following, we will be dealing with solutions to Eq. (26) under nonvanishing boundary conditions. The simplest among them is the CW solution, with constant densities of the three spin components,

$$\begin{aligned} \mathbf{Q}_c &= \mathbf{A}_c e^{i\varphi_c}, \ \mathbf{A}_c \equiv \begin{pmatrix} \beta_c & \alpha_c \\ \alpha_c & -\beta_c \end{pmatrix}, \\ \varphi_c &\equiv kx + \left[2(\alpha_c^2 + \beta_c^2) - k^2\right] t, \end{aligned} \tag{34}$$

where α_c and β_c are real constants, and k is a wave number. In this solution, the constant densities are equal in components $\phi_{\pm 1}$, while the phase difference between the components is π.

Applying the DT to the CW solution $\mathbf{Q}_c$, solving Eqs. (27) for this case, and employing relation (33), we obtain a new family of solutions of Eq. (26) in the form

$$\mathbf{Q}_1 = [\mathbf{A}_c + 4\xi(\mathbf{I} + \mathbf{A}\mathbf{A}^*)^{-1}\mathbf{A}]e^{i\varphi_c}, \tag{35}$$

where the following definitions are used:

$$\mathbf{A} = (\mathbf{\Pi}e^{\theta - i\varphi} + \kappa^{-1}\mathbf{A}_c)(\kappa^{-1}\mathbf{A}_c\mathbf{\Pi}e^{\theta - i\varphi} + \mathbf{I})^{-1}, \tag{36}$$

$$\theta = M_I x + [2\xi M_R - (k + 2\eta)M_I]t, \tag{37}$$

$$\varphi = M_R x - [2\xi M_I + (k + 2\eta)M_R]t, \tag{38}$$

$$M = \sqrt{(k + 2i\lambda)^2 + 4(\alpha_c^2 + \beta_c^2)} \equiv M_{\mathrm{R}} + iM_{\mathrm{I}}, \tag{39}$$

$\kappa \equiv \frac{1}{2}(ik - 2\lambda + iM)$, $\lambda = \xi + i\eta$ is the spectral parameter, and $\mathbf{\Pi} = \begin{pmatrix} \beta & \alpha \\ \alpha & \gamma \end{pmatrix}$ is an arbitrary complex symmetric matrix. Solution (35) reduces back to the CW (34) when $\xi = 0$. It is worth noting that the three-component polar soliton (25) considered in the previous section is not a special example of the solution (35), because the background densities in solution (25) have equal phases in components $\phi_{\pm 1}$.

In particular, with $\mathbf{A}_c = \mathbf{O}$ (zero background), Eq. (35) yields a soliton solution, that can be written as

$$\mathbf{Q}_1 = \frac{4\xi[\mathbf{\Pi}_1 e^{-\theta_s} + (\boldsymbol{\sigma}^y \mathbf{\Pi}_1^* \boldsymbol{\sigma}^y) e^{\theta_s} \det \mathbf{\Pi}_1] e^{i\varphi_s}}{e^{-2\theta_s} + 1 + e^{2\theta_s} |\det \mathbf{\Pi}_1|^2}, \tag{40}$$

where $\theta_s = 2\xi(x - 4\eta t) - \theta_0$, $\varphi_s = 2\eta x + 4(\xi^2 - \eta^2)t$, and θ_0 is an arbitrary real constant which determines the initial position of the soliton, $\boldsymbol{\sigma}^y$ is the Pauli matrix, and $\mathbf{\Pi}_1$ is the polarization matrix [22],

$$\mathbf{\Pi}_1 = \left(2|\alpha|^2 + |\beta|^2 + |\gamma|^2\right)^{-1/2} \mathbf{\Pi} \equiv \begin{pmatrix} \beta_1 & \alpha_1 \\ \alpha_1 & \gamma_1 \end{pmatrix}.$$

Solitons (40) are tantamount to ones found in Ref. [22], where they were classified into the above-mentioned distinct types, the FM and polar ones. Indeed, one can see from expression (40) that, when $\det \mathbf{\Pi}_1 = 0$, the soliton (40) represents a FM state in the spinor model. However, when $\det \mathbf{\Pi}_1 \neq 0$, the solution (40) may have two peaks, corresponding to a polar state (the smaller $\det \mathbf{\Pi}_1$, the larger the distance between the peaks).

We will now produce another particular example of exact solutions (35), that describes the onset and nonlinear development of the MI (modulational instability) of CW states. Indeed, under the conditions $k = 2\eta$ and $\alpha_c^2 + \beta_c^2 > \xi^2$, Eq. (39) yields $M_{\mathrm{I}} = 0$, hence θ is independent of x [see Eq. (37)], and the solution features no localization. Instead, it is x-periodic, through the x-dependence in φ, as given by Eq. (38); in this connection, we notice that

$$M_{\mathrm{R}}^2 = 4\left(\alpha_c^2 + \beta_c^2\right) - 4\xi^2 \tag{41}$$

does not vanish when $M_{\mathrm{I}} = 0$. This spatially periodic state may actually display a particular mode of the nonlinear development of the MI, in an *exact* form. To analyze this issue in detail, we consider a special case with $\mathbf{\Pi} = \epsilon \mathbf{E}$, where $\mathbf{E}$ is a matrix with all elements equal to 1, and ϵ is a small amplitude of the initial perturbation added to a CW to initiate the

onset of the MI. Indeed, the linearization of the initial profile of solution (35) in ϵ yields

$$\mathbf{Q}_1(x,0) \approx [\mathbf{A}_c\rho - \epsilon\chi_1\mathbf{E}\cos(M_Rx) - \epsilon\chi_2\boldsymbol{\sigma}^z\boldsymbol{\sigma}^x\mathbf{A}_c e^{iM_Rx}]\exp(ikx), \tag{42}$$

where $\rho \equiv 1 - \xi(2\xi + iM_R)/(\alpha_c^2 + \beta_c^2)$ with $|\rho| = 1$, $\chi_1 \equiv \xi M_R(2i\xi - M_R)/(\alpha_c^2 + \beta_c^2)$, $\chi_2 \equiv \beta_c\chi_1/(\alpha_c^2 + \beta_c^2)$, $M_R = 2\sqrt{\alpha_c^2 + \beta_c^2 - \xi^2}$, and $\boldsymbol{\sigma}^{z,x}$ are the Pauli matrices. Clearly, the first term in Eq. (42) represents the CW, while the second and third ones are small spatially-modulated perturbations.

Comparing the exact solution (35) obtained in this special case with results of direct numerical simulations of Eqs. (2) with the initial condition (42) (not shown here), one can verify that the numerical solution is very close to the analytical one, both displaying the development of the MI initiated by the small modulational perturbation in the waveform (42). Figure 8 shows the result in terms of the densities of the three components, as provided by the exact solution (35). From this picture we conclude that atoms are periodically transferred, in the spinor BEC, from the spin state $m_F = 0$ into ones $m_F = \pm 1$, and vice versa.

As said above, the exact solution (35) to Eqs. (2) is only valid under the special condition $a = 1$ (i.e., $2g_0 = -g_2$). In the general case ($a \neq 1$), the onset of the MI can be analyzed directly from equations (2) linearized for small perturbations, as it was actually done in Ref. [18]. We will briefly recapitulate the analysis here (in particular, to compare results with the above exact solution). To this end, we take the CW solution (34) (with $k = 0$, which can be fixed by means of the Galilean transformation), and consider its perturbed version,

$$\begin{aligned}
\widetilde{\mathbf{Q}} &= (\mathbf{A}_c + \mathbf{B})\exp[2i\nu(\alpha_c^2 + \beta_c^2)t], \\
\mathbf{B} &= \begin{pmatrix} b_{+1}(x,t) & b_0(x,t) \\ b_0(x,t) & b_{-1}(x,t) \end{pmatrix},
\end{aligned}$$

where $b_{\pm 1}$ and b_0 are weak perturbations. Substituting this expression into Eqs. (2), linearizing them, and assuming the perturbations in the usual form,

$$\begin{aligned}
b_j(x,t) &= b_{j\mathrm{R}}\cos(Kx - \omega t) + ib_{j\mathrm{I}}\sin(Kx - \omega t), \\
j &= +1, 0, -1,
\end{aligned}$$

where $b_{j\mathrm{R,I}}$ are real amplitudes, the linearized equations give rise to two branches of the dispersion relation between the wavenumber K and frequency ω of the perturbation,

$$\omega^2 = K^2(K^2 - 4a\alpha_c^2 - 4a\beta_c^2); \tag{43}$$

$$\omega^2 = K^2(K^2 - 4\nu\alpha_c^2 - 4\nu\beta_c^2). \tag{44}$$

The MI sets in when ω^2 given by either expression ceases to be real and positive. Obviously, this happens if, at least, one condition

$$K^2 < 4a(\alpha_c^2 + \beta_c^2),\ K^2 < 4\nu(\alpha_c^2 + \beta_c^2) \tag{45}$$

holds. Note that these conditions agree with the above exact result, which corresponds to $a = \nu = 1$. Indeed, in that case we may identify $K \equiv M_\mathrm{R}$, according to Eq. (42), and then Eq. (41) entails the constraint, $M_\mathrm{R}^2 < 4\left(\alpha_c^2 + \beta_c^2\right)$, which is identical to conditions (45). Note that the MI may occur in the case of the repulsive spin-independent interactions in Eqs. (2), i.e., if $\nu = -1$, provided that the nonlinear-coupling constant a, which accounts for the strength of the spin-dependent interaction, is positive.

We also explored stability of the soliton solutions (40), of both the polar and FM types, against finite random variations of the coupling constant a in time. The issue is relevant, as the exact solutions are only available for $a = 1$, hence it is necessary to understand if the solitons survive random deviations of a from this special value corresponding to the integrability (in fact, this way one may test the *structural stability* of the solitons generated in the exact form by the inverse-scattering/Darboux transform). Physically, this situation may correspond to a case when the scattering length, that determines the nonlinear coefficient a in Eqs. (2), follows random variations of an external magnetic [29] or laser [30] field which affects the scattering length through the Feshbach resonance. The evolution of the density profiles in the thus perturbed solitons is displayed in Fig. 9. This figure and many other runs of the simulations suggest that the solitons of both the FM and polar types are robust against random changes of a, once they are dynamically stable as exact solutions to the integrable version of the model. Note also that the FM soliton seems more robust than its polar counterpart.

V. CONCLUSION

In this work, we have presented new soliton states in the model based on the coupled NLS equations describing the dynamics of $F = 1$ spinor BECs. In the general (non-integrable) version of the model, several types of elementary exact soliton solutions were reported, which

include one, two, or three components, and represent polar or ferromagnetic (FM) solitons. Their stability was checked by direct simulations, and, in some cases, in an exact analytical form, based on linearized equations for small perturbations. The global stability of the solitons was analyzed by comparing the respective values of the energy for a fixed norm (number of atoms). As a result, we have found stable ground-state solitons of the polar and FM types; stable metastable solitons coexisting with the ground-state ones are possible too, in certain parameter regions.

In the special case of the integrable spinor-BEC model, we have applied the Darboux transformation (DT) to derive a family of exact solutions on top of the CW state. This family includes a solution that explicitly displays full nonlinear evolution of the modulational instability (MI) of the CW state, seeded by a small spatially periodic perturbation. Finally, by means of direct simulations, we have established robustness of one-soliton solutions, of both the FM and polar types, under finite perturbations of the coupling constant. In fact, the latter result implies the structural stability of these solitons.

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Figure Captions

Figure 1. Evolution of the single-component polar soliton (11) with a small random perturbation added, at $t = 0$, to the ϕ_{+1} and ϕ_{-1} components. Parameters are $\nu = 1$, $a = -0.5$, and $\mu = -1$.

Figure 2. The solid, dashed, and dotted curves show, respectively, the evolution of the norms N_0, $N_\pm$ of the ϕ_0 and $\phi_{\pm 1}$ components, and total norm N [see Eqs. (15) and (4)] in the two-component polar soliton (16) perturbed by a small random perturbation introduced in the ϕ_0 component. Parameters are $\nu = 1$, $\mu = -1$, and (a) $a = 1.5$ for $\phi_{+1}\phi_{-1}^* < 0$; (b) $a = 1.5$ for $\phi_{+1}\phi_{-1}^* > 0$; (c) $a = -1.5$ for $\phi_{+1}\phi_{-1}^* < 0$; (d) $a = -1.5$ for $\phi_{+1}\phi_{-1}^* > 0$.

Figure 3. The evolution of the three-component polar soliton (20) under the action of a small random perturbation initially added to the ϕ_0 component. Parameter are $\nu = 1$, $\mu = -0.8$, $\epsilon = 2/\sqrt{5}$, and (a) $a = 0.5$; (b) $a = 1.5$; (c) $a = -0.5$; (d) $a = -1.5$. The meaning of the solid, dashed, and dotted curves is the same as in Fig. 2.

Figure 4. The same as in Fig. 3 for the three-component polar soliton (21). Parameters are $\nu = 1$, $\mu = -0.8$, $\epsilon = 2/\sqrt{5}$, and (a) $a = 0.5$; (b) $a = 1.5$; (c) $a = -0.5$; (d) $a = -1.5$. The meaning of the solid, dashed, and dotted curves is the same as in Figs. 2 and 3.

Figure 5. The evolution of the three-component polar soliton (22) with initially added

random perturbations. Parameters are $\nu = 1$, $\mu_{+1} = -1.21$, $\mu_{-1} = -0.25$, and (a,c) $a = 0.5$, (b,d) $a = -0.5$. In cases (a) and (b), the random perturbation was added to the ϕ_0 component, and in cases (c) and (d) – to components $\phi_{\pm 1}$. The curves marked by 1, 2, 3, and 4 depict, respectively, the evolution of the norms N_{+1}, N_{-1}, N_0 and N, see the definitions in Eqs. (15) and (4).

Figure 6. The evolution of the two-component polar soliton on the CW background, (24), under the action of initial random perturbations added to the ϕ_0 and $\phi_{\pm 1}$ components. Parameters are $\nu = +1, a = -1/2$, and $\mu = -1$.

Figure 7. The same as in Fig. 6, for the finite-background soliton (25). Parameters are $\nu = a = 1$, and $\mu = -0.36$.

Figure 8. The nonlinear development of the modulation instability, as per the exact solution (35) under the conditions $k = 2\eta$ and $\alpha_c^2 + \beta_c^2 > \xi^2$. Parameters are $k = 0.6$, $\eta = 0.3$, $\xi = 1$, $\alpha = \beta = \gamma = e^{-8}$, $\alpha_c = 1.2$, and $\beta_c = 0$.

Figure 9. The evolution of the density distribution in the soliton obtained from solution (35) by adding a randomly time-dependent perturbation of the nonlinear-coupling constant a, with the perturbation amplitude $\pm 5\%$. Parameters are $\nu = a = 1$ (as concerns the unperturbed value of a), $\xi = 1$, $\eta = -0.03$, and $\theta_0 = 0$. (a) The FM soliton, with $\alpha_1 = 0.48$, β_1 and γ_1 being determined by conditions $\alpha_1^2 = \beta_1\gamma_1$ and $2\,|\alpha_1|^2 + |\beta_1|^2 + |\gamma_1|^2 = 1$; (b) the polar soliton, with $\alpha_1 = 0.53$ and $\beta_1 = 0.43$, γ_1 being determined by $2\,|\alpha_1|^2 + |\beta_1|^2 + |\gamma_1|^2 = 1$. The solid and dotted curves show, respectively, the exact solutions and the perturbed ones produced by numerical simulations.